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The evolution of traveling waves in a KPP reaction-diffusion model with cut-off reaction rate. I. Permanent form traveling waves

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Abstract

We consider Kolmogorov-Petrovskii-Piscounov (KPP) type models in the presence of a discontinuous cut-off in reaction rate at concentration $u = u_c$. In Part I, we examine permanent form traveling wave solutions (a companion paper, Part II, is devoted to their evolution in the large time limit). For each fixed cut-off value $0 < u_c < 1$, we prove the existence of a unique permanent form traveling wave with a continuous and monotone decreasing propagation speed $v^*(u_c)$. We extend previous asymptotic results in the limit of small u_c and present new asymptotic results in the limit of large u_c which are, respectively, obtained via the systematic use of matched and regular asymptotic expansions. The asymptotic results are confirmed against numerical results obtained for the particular case of a cut-off Fisher reaction function.

KEYWORDS

asymptotic expansions, permanent form traveling waves, reaction-diffusion equations, singular perturbations

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1 | INTRODUCTION

Traveling waves arise in a wide range of applications in mathematical chemistry and biology (for example, in combustion¹ and in ecology, epidemiology, and genetics^{2,3}). They describe the invasion of chemical or biological reactions and are usually established as a result of the interaction between molecular diffusion, local growth, and saturation. The simplest model that encapsulates this interaction is the Kolmogorov-Petrovskii-Piscounov (KPP) reaction-diffusion equation (also called Fisher-KPP equation^{4,5}). In one spatial dimension, this describes the evolution of the concentration $u(x, t)$ as

$$u_t = u_{xx} + f(u), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (1a)$$

with $u(x, 0) = u_0(x)$, where $u_0 : \mathbb{R} \rightarrow \mathbb{R}$ is piecewise continuous and smooth with limits 0 and 1 as $x \rightarrow \infty$ and $x \rightarrow -\infty$, respectively. This is typically supplemented with boundary conditions

$$u(x, t) \rightarrow \begin{cases} 1, & \text{as } x \rightarrow -\infty \\ 0, & \text{as } x \rightarrow \infty \end{cases} \quad (1b)$$

with these limits being uniform for $t \in [0, T]$ and any $T > 0$. The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is a normalized KPP-type reaction function which satisfies conditions that $f \in C^1(\mathbb{R})$ and

$$f(0) = f(1) = 0, \quad f'(0) = 1, \quad f'(1) < 0 \quad (2a)$$

and in addition

$$0 < f(u) \leq u \quad \forall u \in (0, 1), \quad f(u) < 0 \quad \forall u \in (1, \infty). \quad (2b)$$

A prototypical example of such a KPP reaction function is the Fisher reaction function⁴ given by

$$f(u) = u(1 - u). \quad (3a)$$

An illustration of $f(u)$ against u is given in Figure 1A. Another popular example of a KPP reaction function is

$$f(u) = u(1 - u^2). \quad (3b)$$

It is well-known^{2,5-7} that the initial-boundary value problem (1) for the KPP equation supports a one-parameter family of nonnegative permanent form traveling wave solutions of the form

$$u(x, t) = U(y) = U(x - vt) \quad \forall (x, t) \in \mathbb{R} \times \mathbb{R}^+. \quad (4)$$

These remain steady in time in a reference frame moving in the positive x direction with speed $v \geq 0$ to be determined. Their existence and uniqueness (up to linear translation in origin of the independent coordinate y) is established for

$$v \geq v_m = 2, \quad (5)$$

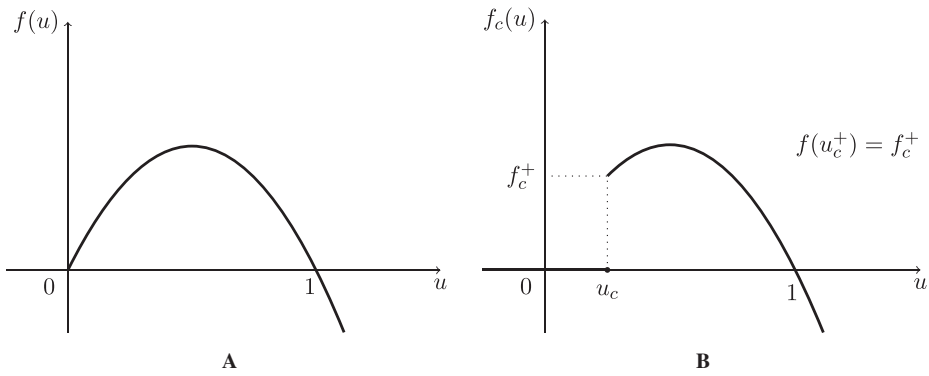


FIGURE 1 A, A sketch of a KPP-type reaction function. B, A sketch of a cut-off KPP-type reaction function

where v_m denotes the minimum speed of propagation. This is achieved by analyzing the following nonlinear boundary value problem, namely,

$$U'' + vU' + f(U) = 0, \quad -\infty < y < \infty, \quad (6a)$$

$$U(y) \geq 0, \quad -\infty < y < \infty, \quad (6b)$$

$$U(y) \rightarrow \begin{cases} 1, & \text{as } y \rightarrow -\infty \\ 0, & \text{as } y \rightarrow \infty, \end{cases} \quad (6c)$$

where the dash denotes differentiation with respect to y . This is obtained by inserting (4) into Equation (1a) and using (1b) together with the initial conditions. The analysis is based on examining the existence of a unique heteroclinic orbit connecting the stable fixed point $(U, U') = (0, 0)$ to the unstable fixed point $(U, U') = (1, 0)$ in the (U, U') phase plane of the equivalent two-dimensional dynamical system obtained from (6). It is also used to establish that $U(y)$ is monotone decreasing in $y \in \mathbb{R}$. When translational invariance is fixed by requiring that $U(0) = 1/2$, then explicit expressions for the behavior of the permanent form traveling wave near the two fixed points are given by

$$U(y) \sim \begin{cases} (A_\infty y + B_\infty) e^{-y}, & \text{as } y \rightarrow \infty, v = v \geq v_m = 2 \\ C_\infty e^{\alpha(v)y}, & \text{as } y \rightarrow \infty, v > v \geq v_m = 2 \end{cases} \quad (7a)$$

and for all $v \geq v_m = 2$,

$$U(y) \sim 1 - A_{-\infty} e^{\gamma(v)y}, \quad \text{as } y \rightarrow -\infty, \quad (7b)$$

where

$$\alpha(v) = \frac{1}{2}(-v + \sqrt{v^2 - 4}) < 0, \quad \gamma(v) = 1/2(-v + \sqrt{v^2 + 4|f'(1)|}) > 0, \quad (7c)$$

with $A_\infty(>0)$, B_∞ , $C_\infty(>0)$, and $A_{-\infty}(>0)$ being globally determined constants, dependent on the nonlinearity of the boundary value problem (6) (see, for example, Refs. 2, 8).

A key result is that the initial condition in $u_0(x)$ determines the permanent form traveling wave solution that emerges at large times. When $u_0(x)$ is sufficiently close to a Heaviside function, specifically, $u_0(x) \leq O(e^{-x})$ (meaning $O(e^{-x})$ or $o(e^{-x})$) as $x \rightarrow \infty$, the solution to the KPP initial-boundary value problem (1) converges at large times to the permanent form traveling solution with minimum speed $v_m = 2$ (see, for example, Refs. 5, 6, 9, 10) at an algebraic rate determined in Refs. 11–13. The mechanism which selects the speed of propagation of the emerging permanent form traveling wave solution (as well as the rate of convergence) is based on the linearization of the KPP equation (1a) at the leading edge of the traveling wave. There, the concentration is small and the dynamics are unstable. As a result, any modification of the dynamics near the leading edge of the traveling wave would invalidate this speed selection mechanism.

This is precisely the case for the cut-off KPP model that Brunet and Derrida¹⁴ proposed and considered. Motivated by the discrete nature of chemical and biological phenomena at the microscopic level, they took a reaction function that is effectively deactivated at points where the concentration u lies at or below a threshold value $u_c \in (0, 1)$. This case corresponds to the cut-off KPP equation given by

$$u_t = u_{xx} + f_c(u), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (8a)$$

with $u(x, 0) = u_0(x)$, which is once more supplemented with the boundary conditions

$$u(x, t) \rightarrow \begin{cases} 1, & \text{as } x \rightarrow -\infty \\ 0, & \text{as } x \rightarrow \infty \end{cases} \quad (8b)$$

uniformly for $t \in [0, T]$ for all $T > 0$. The main difference is that the reaction function $f : \mathbb{R} \rightarrow \mathbb{R}$ in the KPP equation (1) is replaced with a cut-off reaction function $f_c : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f_c(u) = \begin{cases} f(u), & u \in (u_c, \infty), \\ 0, & u \in (-\infty, u_c], \end{cases} \quad (8c)$$

where $f(u)$ satisfies the KPP conditions (2). An illustration of $f_c(u)$ against u is given in Figure 1B, with $f_c^+ = f_c(u_c^+)$ where $f_c(u_c^+)$ is the short notation for $\lim_{u \rightarrow u_c^+} f_c(u)$. We remark that $f_c(u)$ exhibits similarities with reaction functions arising in models of combustion in which u_c represents an ignition temperature threshold^{1,15}. Focussing on the initial conditions

$$u_0(x) = \begin{cases} 1, & \text{for } x < 0, \\ 0, & \text{for } x \geq 0, \end{cases} \quad (8d)$$

we henceforth refer to this initial-boundary value problem as IVP. Brunet and Derrida¹⁴ proposed (8) as a model of front propagation arising in discrete systems of interacting particles. Such systems are for example, lattice models with discrete particles which make diffusive hops to neighboring sites, and which have some birth-death type of reaction¹⁶. In the continuum limit, obtained by allowing an arbitrarily large number of particles per lattice site, Brunet and Derrida¹⁴ conjectured that discreteness in concentration values can be represented by an effective cut-off u_c where u_c may be viewed as the effective mass of a single particle. The idea is that for $u < u_c$, diffusion dominates over growth. Although the connection between (8) and discrete systems of interacting

particles is phenomenological, model (8) remains useful in providing insight into their behavior. Analyzing the specific example (3b), Brunet and Derrida¹⁴ considered the behavior of permanent form traveling wave solutions for small values of u_c . Their main result is a prediction for the propagation speed $v^*(u_c)$ of the *unique* permanent form traveling wave given by

$$v^*(u_c) \approx 2 - \frac{\pi^2}{(\ln u_c)^2}, \quad \text{as } u_c \rightarrow 0^+, \quad (9)$$

which they obtained using a two-region informal point patching procedure (see also Ref. 17 where (9) was compared against numerical simulations of lattice particle models). This significant result demonstrates the strong influence of a cut-off on the value of $v^*(u_c)$ for small values of u_c . The same approximation to $v^*(u_c)$ has also been obtained via an alternative variational approach in Refs. 18, 19. Subsequently, a more rigorous approach was employed by Dumortier et al²⁰ who used geometric desingularization, to prove the existence and uniqueness of a permanent form traveling wave with

$$v^*(u_c) \sim 2 - \frac{\pi^2}{(\ln u_c)^2} + O\left(\frac{1}{|\ln u_c|^3}\right), \quad \text{as } u_c \rightarrow 0^+. \quad (10)$$

All these results have restricted validity to the small u_c limit with specific choices of cut-off KPP-type reaction function (8d), the most common based on $f(u)$ given by (3b)¹. Expression (10) was found in Ref. 20 to be generic when considering a slightly more general class of cut-off KPP-type reaction functions, namely, identical to (8d) when $u \in (u_c, \infty)$ but has $f_c(u) = o(1)$ uniformly for $u \in [0, u_c]$ as $u_c \rightarrow 0^+$.

There are a number of fundamental questions that remain. The first question concerns the existence and uniqueness of a permanent form traveling wave solution for *arbitrary* threshold values u_c and KPP reaction functions $f(u)$. The second question concerns the propagation speed of such permanent form traveling wave solutions for arbitrary threshold values u_c . The third question is with regard to the shape of the permanent form traveling wave solution. The fourth question concerns a systematic approach that captures the leading as well as higher-order corrections to the asymptotic behavior of the speed and shape of the permanent form traveling wave solution as $u_c \rightarrow 0^+$ and $u_c \rightarrow 1^-$. The second limit may be less relevant for discrete systems of interacting particles. It is however relevant in models of combustion since the ignition temperature that determines the cut-off is not necessarily small¹. A final question concerns the evolution in time to the permanent form traveling wave solution via the initial boundary value problem IVP. Part I of this series of papers addresses the first four of these questions while part II addresses the fifth and last question. In particular, we study classical solutions $u : \mathbb{R} \times \overline{\mathbb{R}}^+ \rightarrow \mathbb{R}$ to IVP for the cut-off KPP equation (8). In this paper, we proceed as follows. In Section 2, we reformulate IVP as a moving boundary problem. We then make a simple coordinate transformation to consider an equivalent initial-boundary value problem that we refer to as QIVP. In Section 3, we examine the possibility that QIVP supports permanent form traveling wave solutions $U_T(y) = U_T(x - vt)$ where $U_T \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ satisfies the nonlinear boundary value problem,

$$U_T'' + vU_T' + f_c(U_T) = 0, \quad y \in \mathbb{R} \setminus \{0\}, \quad (11a)$$

¹ There have been a number of results obtained for other cut-off reaction functions (see, for example, Refs. 15, 21–23), but we focus on the cut-off KPP-type reaction functions.

$$U_T \geq u_c \quad \forall y < 0, \quad 0 \leq U_T \leq u_c, \quad \forall y > 0, \quad (11b)$$

$$U_T(0) = u_c, \quad (11c)$$

$$U_T(y) \rightarrow \begin{cases} 1, & \text{as } y \rightarrow -\infty \\ 0, & \text{as } y \rightarrow \infty. \end{cases} \quad (11d)$$

We establish the following theorem.

Theorem 1. *For each fixed $u_c \in (0, 1)$, QIVP has a unique permanent form traveling wave solution $U_T : \mathbb{R} \rightarrow \mathbb{R}$, with the propagation speed given by $v^*(u_c)$. Here, $v^* : (0, 1) \rightarrow \mathbb{R}^+$ is continuous and monotone decreasing, with*

$$v^*(u_c) \rightarrow \begin{cases} 0, & \text{as } u_c \rightarrow 1^-, \\ 2, & \text{as } u_c \rightarrow 0^+, \end{cases}$$

where 2 is the minimum propagation speed of the permanent form traveling wave solution in the absence of cut-off ($u_c = 0$). In addition, $U_T(y)$ is strictly monotone decreasing for $y \in \mathbb{R}$, with $U_T(0) = u_c$, and

$$U_T''(0^+) - U_T''(0^-) = -f_c^+, \quad (12a)$$

$$U_T(y) = u_c e^{-v^*(u_c)y} \quad \forall y \in \mathbb{R}^+, \quad (12b)$$

$$U_T(y) \sim 1 - A_{-\infty} e^{\lambda_+(v^*(u_c))y} \quad \text{as } y \rightarrow -\infty, \quad (12c)$$

for some global constant $A_{-\infty} > 0$ (which depends upon u_c), and

$$\lambda_+(v) = \frac{1}{2} \left(-v + \sqrt{v^2 + 4|f'_c(1)|} \right) > 0.$$

Furthermore,

$$v^*(u_c) \sim |f'_c(1)|^{\frac{1}{2}} (1 - u_c) \quad \text{as } u_c \rightarrow 1^-. \quad (13)$$

In Sections 4 and 5, we use matched asymptotic expansions to develop the detailed asymptotic structure to the permanent form traveling wave solutions as $u_c \rightarrow 0^+$ and as $u_c \rightarrow 1^-$, respectively. These are used to obtain higher-order corrections to (10) and (13) in a systematic manner. In the

first limit, the analysis is carried out on the direct problem (rather than the phase plane). It highlights that higher-order corrections are controlled by two global constants A_∞ and B_∞ associated with the leading behaviour of U_m , the permanent form traveling wave solution to the non cut-off problem (1) with minimal speed $v = v_m$ (see equation (7a)). These global constants represent the nonlinearity in the problem when u_c is small. The analysis is readily generalized to degenerate and singular KPP conditions, obtained for example when $f'(0) = 0$ or $f(u) \sim u^{1/2}$ as $u \rightarrow 0^+$, respectively. Section 6 presents numerical examples for the specific Fisher cut-off reaction function (3a). The paper concludes with a discussion in Section 7.

2 | FORMULATION OF EVOLUTION PROBLEM QIVP

Due to the discontinuity in $f_c(u)$ at $u = u_c$, it is convenient to restructure IVP as a moving boundary problem. To this end, we introduce the domains:

$$\begin{aligned} D^L &= \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ : x < s(t)\}, \\ D^R &= \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ : x > s(t)\}, \end{aligned} \quad (14a)$$

and the curve

$$\mathcal{L} = \{(x, t) \in \mathbb{R} \times \mathbb{R}^+ : x = s(t)\}, \quad (14b)$$

that describes the moving boundary between the two domains. The boundary is expressed in terms of $s(t)$ which satisfies $u(s(t), t) = u_c$, with $u \geq u_c$ in $\overline{D^L}$ and $u \leq u_c$ in $\overline{D^R}$. In this context, a classical solution will have $u : \mathbb{R} \times \overline{\mathbb{R}^+} \rightarrow \mathbb{R}$ and $s : \overline{\mathbb{R}^+} \rightarrow \mathbb{R}$ such that,

$$u \in C(\mathbb{R} \times \overline{\mathbb{R}^+} \setminus \{(0, 0)\}) \cap C^{1,1}(\mathbb{R} \times \mathbb{R}^+) \cap C^{2,1}(D^L \cup D^R), \quad s \in C^1(\mathbb{R}^+), \quad (15a)$$

$$s(0^+) = 0. \quad (15b)$$

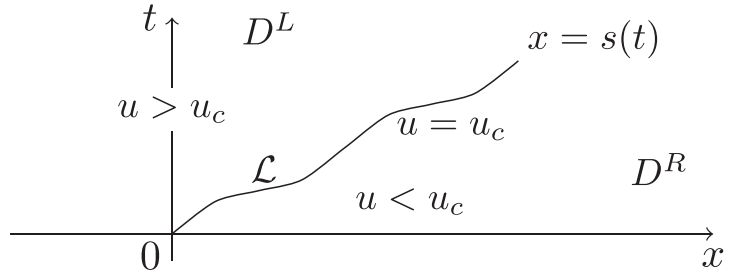
The moving boundary problem is then formulated as follows:

$$u_t = u_{xx} + f_c(u), \quad (x, t) \in D^L \cup D^R, \quad (16a)$$

$$u \geq u_c \text{ in } \overline{D^L}, \quad u \leq u_c \text{ in } \overline{D^R}, \quad (16b)$$

$$u(x, 0) = \begin{cases} 1, & \text{for } x < 0, \\ 0, & \text{for } x \geq 0, \end{cases} \quad (16c)$$

FIGURE 2 A sketch of the evolution of the boundary $u = u_c$ in the moving boundary problem



$$u(x, t) = \begin{cases} 1, & \text{as } x \rightarrow -\infty \\ 0, & \text{as } x \rightarrow \infty \end{cases} \quad (16d)$$

uniformly for $t \in [0, T]$ for all $T > 0$ and

$$u(s(t), t) = u_c, \quad u_x(s(t)^+, t) = u_x(s(t)^-, t), \quad t \in \mathbb{R}^+, \quad (16e)$$

$$s(0^+) = 0. \quad (16f)$$

The situation is illustrated in Figure 2. It is now convenient to make the simple coordinate transformation $(x, t) \rightarrow (y, t)$ with $y = x - s(t)$. We then introduce the following domains:

$$Q^L = \mathbb{R}^- \times \mathbb{R}^+, \quad Q^R = \mathbb{R}^+ \times \mathbb{R}^+, \quad (17)$$

with $u : \mathbb{R} \times \overline{\mathbb{R}^+} \rightarrow \mathbb{R}$ and $s : \overline{\mathbb{R}^+} \rightarrow \mathbb{R}$ such that

$$u \in C(\mathbb{R} \times \overline{\mathbb{R}^+} \setminus \{(0, 0)\}) \cap C^{1,1}(\mathbb{R} \times \mathbb{R}^+) \cap C^{2,1}(Q^L \cup Q^R), \quad s \in C^1(\mathbb{R}^+). \quad (18)$$

The equivalent problem to (16) is then given by

$$u_t - \dot{s}(t)u_y = u_{yy} + f_c(u), \quad (y, t) \in Q^L \cup Q^R, \quad (19a)$$

$$u \geq u_c \text{ in } \overline{Q}^L, \quad u \leq u_c \text{ in } \overline{Q}^R, \quad (19b)$$

$$u(y, 0) = \begin{cases} 1, & y < 0, \\ 0, & y \geq 0, \end{cases} \quad (19c)$$

$$u(y, t) \rightarrow \begin{cases} 1, & \text{as } y \rightarrow -\infty, \\ 0, & \text{as } y \rightarrow \infty, \end{cases} \quad (19d)$$

uniformly for $t \in [0, T]$ for all $T > 0$ and

$$u(0, t) = u_c, \quad u_y(0^+, t) = u_y(0^-, t), \quad t \in \mathbb{R}^+, \quad (19e)$$

$$s(0^+) = 0, \quad (19f)$$

where the dot denotes differentiation with respect to time, t . This initial-boundary value problem will henceforth be referred to as QIVP. On using the classical maximum principle and comparison theorem (see, for example, Refs. 2 and 24), together with translational invariance in y , and the regularity in (18), we can establish the following basic qualitative properties concerning QIVP, namely,

$$0 < u(y, t) < u_c \quad \forall (y, t) \in Q^R, \quad u_c < u(y, t) < 1 \quad \forall (y, t) \in Q^L, \quad (20a)$$

$$u(y, t) \text{ is strictly monotone decreasing in } y \in \mathbb{R} \quad \forall t \in \mathbb{R}^+. \quad (20b)$$

In addition, via the partial differential equation (19a) and the regularity conditions (18), we have

$$\begin{aligned} \lim_{y \rightarrow 0^+} u_{yy}(y, t) &= \lim_{y \rightarrow 0^+} (u_t(y, t) - \dot{s}(t)u_y(y, t)) \\ &= -\dot{s}(t)u_y(0, t) \quad \forall t \in \mathbb{R}^+, \end{aligned} \quad (20c)$$

$$\begin{aligned} \lim_{y \rightarrow 0^-} u_{yy}(y, t) &= \lim_{y \rightarrow 0^-} (u_t(y, t) - \dot{s}(t)u_y(y, t) - f(u(y, t))) \\ &= -\dot{s}(t)u_y(0, t) - f_c^+ \quad \forall t \in \mathbb{R}^+, \end{aligned} \quad (20d)$$

with the limits in (20c) and (20d) being uniform for $t \in [t_0, t_1]$ (for any $0 < t_0 < t_1$). It follows from (20c) and (20d) that

$$[u_{yy}(y, t)]_{y=0^-}^{y=0^+} = f_c^+ \quad \forall t \in \mathbb{R}^+, \quad (20e)$$

while, using (20b), (20d), and the regularity condition (18), we establish that

$$u_y(y, t) < 0 \quad \forall (y, t) \in \mathbb{R} \times \mathbb{R}^+. \quad (20f)$$

The remainder of this paper and its companion (part II) concentrates on the analysis of QIVP. Specifically, in this paper we consider the existence and uniqueness of permanent form traveling wave solutions to QIVP including their asymptotic behavior in the limits of $u_c \rightarrow 0^+$ and $u_c \rightarrow 1^-$ via the method of matched and regular asymptotic expansions.

3 | PERMANENT FORM TRAVELING WAVES IN QIVP

We anticipate that as $t \rightarrow \infty$, a permanent form traveling wave solution will develop in the solution to QIVP, advancing with a (nonnegative) propagation speed, allowing for the transition between the fully reacted state, $u = 1$ as $y \rightarrow -\infty$, to the unreacted state, $u = 0$ as $y \rightarrow \infty$. Therefore, in this section we focus attention on the possibility of QIVP supporting permanent form traveling wave solutions (henceforth referred to as PTW solutions). We begin by establishing the existence and uniqueness of a PTW to QIVP for each fixed $u_c \in (0, 1)$, denoting the unique propagation speed by $v = v^*(u_c)$. We then consider limiting values of $v^*(u_c)$ as $u_c \rightarrow 0^+$ and $u_c \rightarrow 1^-$. The results established in this section provide proof of Theorem 1 as stated in Section 1.

3.1 | The existence and uniqueness of a PTW solution to QIVP

A PTW solution to QIVP, with constant speed of propagation $v \geq 0$, is a steady-state solution to QIVP with $u : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ and $s : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$u(y, t) = U_T(y) \quad \forall (y, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (21)$$

$$\dot{s}(t) = v \quad \forall t \in \mathbb{R}^+, \quad (22)$$

where $U_T \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ and $v \geq 0$ satisfy the nonlinear boundary value problem,

$$U_T'' + vU_T' + f_c(U_T) = 0, \quad y \in \mathbb{R} \setminus \{0\}, \quad (23a)$$

$$U_T \geq u_c \quad \forall y < 0, \quad 0 \leq U_T \leq u_c \quad \forall y > 0, \quad (23b)$$

$$U_T(0) = u_c, \quad (23c)$$

$$U_T(y) \rightarrow \begin{cases} 1, & \text{for } y \rightarrow -\infty, \\ 0, & \text{for } y \rightarrow \infty, \end{cases} \quad (23d)$$

where the dash denotes differentiation with respect to y . The nonlinear boundary value problem (23) can be thought of as a nonlinear eigenvalue problem with the eigenvalue being the propagation speed $v \geq 0$.

It is convenient to consider the ordinary differential equation (23) as the following equivalent autonomous first-order two-dimensional dynamical system, with $\alpha = U_T$ and $\beta = U_T'$, namely,

$$\begin{aligned} \alpha' &= \beta, \\ \beta' &= -v\beta - f_c(\alpha). \end{aligned} \quad (24)$$

We will analyze this dynamical system in the (α, β) phase plane for $v \geq 0$. In particular, it is straightforward to establish that the existence of a solution to (23) is equivalent to the existence of a heteroclinic connection in the (α, β) phase plane, for the dynamical system (24), which connects the equilibrium point $(1, 0)$, as $y \rightarrow -\infty$, to the equilibrium point $(0, 0)$, as $y \rightarrow \infty$ (the translational invariance is then fixed by condition (23c) which requires that $\alpha(0) = u_c$). From (23b), this heteroclinic connection must remain in the $\alpha \geq 0$ half plane of the (α, β) phase plane, which we denote by $R^+ = \{(\alpha, \beta) : (\alpha, \beta) \in \overline{\mathbb{R}}^+ \times \mathbb{R}\}$. We henceforth focus on this region of the (α, β) phase plane.

However, before we proceed further, it is first worth considering the effect of introducing the cut-off into the reaction function on the dynamical system (24). To that end, we introduce the function $\vec{Q} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where $\vec{Q}(\alpha, \beta)$ is given by

$$\vec{Q}(\alpha, \beta) = (\beta, -v\beta - f_c(\alpha)), \quad (25)$$

to represent the vector field generating the dynamical system (24). We observe that, in the (α, β) phase plane, the effect of the discontinuity in $f_c(\alpha)$ across the line $\alpha = u_c$ is simply to *refract* the phase paths passing through this line. In particular, for each $\beta \in \mathbb{R}$, there is exactly one phase path passing through (u_c, β) , which has tangent vectors, $\vec{Q}(u_c^-, \beta) = (\beta, -v\beta)$ and $\vec{Q}(u_c^+, \beta) = (\beta, -v\beta - f_c^+)$. Thus, the refraction vector for the phase paths which cross the line $\alpha = u_c$ is

$$\vec{R}(u_c, \beta) = \vec{Q}(u_c^+, \beta) - \vec{Q}(u_c^-, \beta) = (0, -f_c^+). \quad (26)$$

We observe that the refraction vector (26) is independent of $(\beta, v) \in \mathbb{R} \times \overline{\mathbb{R}}^+$ and depends continuously on $u_c \in (0, 1)$. It follows that

$$\vec{R}(u_c, \beta) \rightarrow \vec{0} \quad \text{as} \quad u_c \rightarrow 0, \quad (27)$$

uniformly in $(\beta, v) \in \mathbb{R} \times \overline{\mathbb{R}}^+$. After determining the effect of the discontinuity on the phase paths of the dynamical system (24) in R^+ , we next consider the equilibrium points of (24) in R^+ . These are readily found to be at locations

$$\vec{e}_a = (a, 0) \quad \text{for each } a \in [0, u_c], \quad \vec{e}_1 = (1, 0). \quad (28)$$

We begin by examining the local phase portrait in the neighborhood of the equilibrium point \vec{e}_1 . We find that \vec{e}_1 is a hyperbolic equilibrium point. Moreover, \vec{e}_1 is a saddle point with eigenvalues

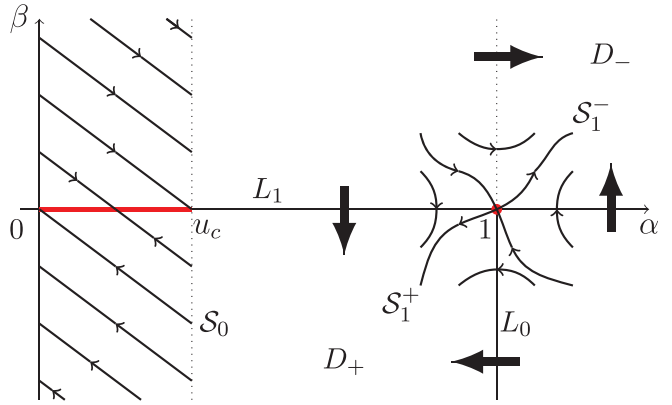
$$\lambda_{\pm}(v) = \frac{1}{2} \left(-v \pm \sqrt{v^2 + 4|f'_c(1)|} \right). \quad (29)$$

The associated local one-dimensional unstable and stable manifolds of \vec{e}_1 are, respectively, given by

$$\beta(\alpha) = -\lambda_{\pm}(v)(1 - \alpha). \quad (30)$$

We denote the phase path which forms the part of the (one-dimensional) unstable manifold entering $D_+ = \{(\alpha, \beta) : 0 < \alpha < 1, \beta < 0\}$ as S_1^+ . Similarly, we denote as S_1^- the phase path which forms part of the (one-dimensional) unstable manifold entering $D_- = \{(\alpha, \beta) : \alpha > 1, \beta > 0\}$. The

FIGURE 3 The local phase portrait for the equilibrium points of the dynamical system (24). The thick black arrows denote the direction of the vector field $\vec{Q}(\alpha, \beta)$ (see (25)) along the line segments L_0, L_1 and on the boundary of D_-



situation is illustrated in Figure 3. We next determine the local phase portrait of the equilibrium points \vec{e}_a for each $a \in [0, u_c]$. For $a \in (0, u_c)$ and $v > 0$, each of the equilibrium points \vec{e}_a is nonhyperbolic with a single (one-dimensional) stable manifold in R^+ given by $\{(\alpha, \beta) : \beta = -v(\alpha - a); 0 \leq \alpha \leq u_c\}$. Also, the equilibrium point \vec{e}_0 is nonhyperbolic with a single (one-dimensional) stable manifold in R^+ which we will denote by

$$S_0 = \{(\alpha, \beta) : \beta = -v\alpha; 0 \leq \alpha \leq u_c\}. \quad (31)$$

Finally, the equilibrium point \vec{e}_{u_c} is again nonhyperbolic, and, for $0 \leq \alpha \leq u_c$, has a single (one-dimensional) stable manifold in R^+ given by $\{(\alpha, \beta) : \beta = -v(\alpha - u_c); 0 \leq \alpha \leq u_c\}$. In fact, the collection of phase paths of the dynamical system (27) in the region $\{(\alpha, \beta) : 0 \leq \alpha \leq u_c, \beta \leq 0\}$ is given by the family of curves $\beta = c - v\alpha$, for each $c \in \mathbb{R}$. This is illustrated in Figure 3. Next, for the line segment $\{(\alpha, \beta) : \alpha = 1, \beta > 0\}$, we observe the following:

$$\vec{Q}(\alpha, \beta) \cdot (1, 0) = \beta > 0. \quad (32)$$

Similarly, for the line segment $\{(\alpha, \beta) : \alpha > 1, \beta = 0\}$, we observe that

$$\vec{Q}(\alpha, \beta) \cdot (0, 1) = -f_c(\alpha) > 0. \quad (33)$$

Together with the local structure at the equilibrium point \vec{e}_1 , we conclude from (32) and (33) that the region D_- is a strictly positively invariant region for the dynamical system (24). We now examine the line segments $L_0 = \{(\alpha, \beta) : \alpha = 1, \beta < 0\}$ and $L_1 = \{(\alpha, \beta) : u_c < \alpha < 1, \beta = 0\}$, we observe that

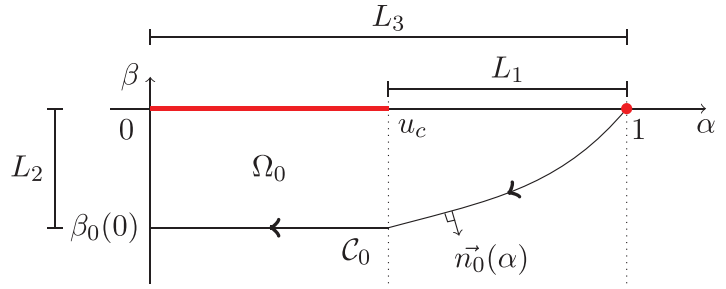
$$\vec{Q}(\alpha, \beta) \cdot (-1, 0) = -\beta > 0 \quad \forall (\alpha, \beta) \in L_0, \quad \vec{Q}(\alpha, \beta) \cdot (0, -1) = f_c(\alpha) > 0 \quad \forall (\alpha, \beta) \in L_1. \quad (34)$$

In addition, for $v > 0$, we observe that for all $(\alpha, \beta) \in R^+$

$$\nabla \cdot \vec{Q}(\alpha, \beta) = -v < 0. \quad (35)$$

Thus, for any $v > 0$, it follows from the Bendixson negative criterion (see, for example, Ref. 25) that (27) has no periodic orbits, homoclinic orbits, or heteroclinic cycles in R^+ . Finally, we observe that

FIGURE 4 The phase path C_0 which forms part of the unstable manifold of the equilibrium point $\vec{e}_1 = (1, 0)$ of the dynamical system (24) when $v = 0$



at each $(\alpha, \beta) \in R^+ \setminus (\{\vec{e}_1\} \cup \{\vec{e}_a : 0 \leq a \leq u_c\})$ the vector field $\vec{Q}(\alpha, \beta)$ rotates continuously clockwise for increasing $v \geq 0$. At the equilibrium point \vec{e}_1 , the unstable manifold S_1^+ rotates clockwise for increasing $v \geq 0$, as does the stable manifold S_0 at the equilibrium point \vec{e}_0 . As the phase path S_1^- enters D_- on leaving \vec{e}_1 , and we have established that D_- is a strictly positively invariant region for the dynamical system (24), we conclude that this cannot correspond to a heteroclinic connection between \vec{e}_1 and \vec{e}_0 . Thus, at any $v \geq 0$, the existence of a heteroclinic connection in R^+ connecting \vec{e}_1 , as $y \rightarrow -\infty$, to \vec{e}_0 , as $y \rightarrow \infty$, is equivalent to the phase path S_1^+ , leaving \vec{e}_1 , being coincident with the phase path S_0 , entering \vec{e}_0 . It also follows that, at those $v \geq 0$ when such a heteroclinic connection exists, then it is unique.

We are now in a position to investigate for which values of $v \geq 0$, if any, the required heteroclinic connection exists in R^+ . When $v = 0$, it follows directly from (24) that the phase path S_1^+ has graph $(\alpha, \beta_0(\alpha))$ where

$$\beta_0(\alpha) = -\left(2 \int_{\alpha}^1 f_c(\gamma) d\gamma\right)^{\frac{1}{2}}, \quad (36)$$

for $\alpha \in [0, 1]$. Thus, $\beta_0(\alpha)$ is (non-positive) non-decreasing for $\alpha \in [0, 1]$ with

$$\beta_0(0) = -\left(2 \int_{u_c}^1 f_c(\gamma) d\gamma\right)^{\frac{1}{2}} < 0 \quad \text{and} \quad \beta_0'(1) = (-f_c'(1))^{\frac{1}{2}}. \quad (37)$$

We also note that $\beta_0(\alpha)$ is continuous and differentiable except for a jump in derivative at $\alpha = u_c$ when $\beta_0'(u_c^+) = -f_c^+/ \beta_0(0)$ while $\beta_0'(u_c^-) = 0$.

We denote the phase path $S_1^+|_{v=0}$ as C_0 , and note from (36) that $C_0 \subset \bar{D}^+$ as illustrated in Figure 4. We conclude from (37) that when $v = 0$ no heteroclinic connection exists from \vec{e}_1 to \vec{e}_0 . Moreover, it follows from the rotational properties of the vector field $\vec{Q}(\alpha, \beta)$ with increasing $v \geq 0$, as discussed earlier, that, for each $v > 0$, we have

$$\vec{Q}(\alpha, \beta_0(\alpha)) \cdot \vec{n}_0(\alpha) < 0, \quad (38)$$

for all $\alpha \in [0, 1)$, where $\vec{n}_0(\alpha)$ is the unit normal to C_0 for $\alpha \in (u_c, 1]$ as shown in Figure 4. We define the line segments $L_2 = \{(\alpha, \beta) : \alpha = 0, \beta_0(0) < \beta < 0\}$ and $L_3 = \{(\alpha, \beta) : 0 \leq \alpha \leq 1, \beta = 0\}$ and denote the region $\Omega_0 \subset D_+$ as that region bounded by $\partial\Omega_0 = L_2 \cup L_3 \cup C_0$. We observe, via the rotational properties of S_1^+ at \vec{e}_1 with increasing $v \geq 0$, that for any $v > 0$, $S_1^+|_v$ enters Ω_0 on leaving \vec{e}_1 . Moreover, we recall, $\bar{\Omega}_0$ contains no periodic orbits, homoclinic orbits, or heteroclinic

cycles. It then follows from (34), (38), and the Poincaré-Bendixson Theorem (see, for example, Ref. 25), that $S_1^+|_v$ must leave Ω_0 through L_2 (at finite y) or connect with \vec{e}_a , for some $a \in [0, u_c]$ (as $y \rightarrow \infty$). For each $v \geq 0$, this observation allows us to classify the behavior of $S_1^+|_v$, by introducing the following function $\bar{s} : \overline{\mathbb{R}}^+ \rightarrow \mathbb{R}$, such that,

$\bar{s}(v)$ = The distance, measured from the origin of the (α, β) plane, to the point of intersection of $S_1^+|_v$ with L_2 (negative distance) or L_3 (positive distance).

We have immediately that

$$\bar{s}(0) = \beta_0(0) < 0, \quad \text{and} \quad \beta_0(0) < \bar{s}(v) \leq u_c, \quad (39)$$

for all $v > 0$. Moreover, since $\vec{Q}(\alpha, \beta)$ depends continuously on $(\alpha, \beta, v) \in \overline{D}_+ \times \overline{\mathbb{R}}^+ \setminus \{(\beta, u_c) : \beta \leq 0\} \times \overline{\mathbb{R}}^+$, the refraction vector (26) for phase paths crossing the line $\alpha = u_c$ in D_+ is independent of $(\beta, v) \in \mathbb{R}^- \times \overline{\mathbb{R}}^+$, and $\overline{\Omega}_0$ is compact, we may conclude that $\bar{s} \in C(\overline{\mathbb{R}}^+)$. In addition, from the rotational properties of the vector field $\vec{Q}(\alpha, \beta)$ in R^+ with increasing $v \geq 0$, we deduce that $\bar{s}(v_2) > \bar{s}(v_1) \forall v_2 > v_1 \geq 0$. Therefore, $\bar{s} : \overline{\mathbb{R}}^+ \rightarrow \mathbb{R}$ is a continuous and strictly monotone increasing function. Next, take

$$v > v_c(u_c) = \left(\frac{1}{u_c} \sup_{\gamma \in (u_c, 1]} f_c(\gamma) \right)^{\frac{1}{2}}. \quad (40)$$

Then, with $\beta_c = -vu_c$, we have

$$\vec{Q}(\alpha, \beta_c) \cdot (0, 1) = v^2 u_c - f_c(\alpha) > \sup_{\gamma \in (u_c, 1]} f_c(\gamma) - f_c(\alpha) \geq 0, \quad (41)$$

for all $\alpha \in (u_c, 1]$, and recall that $S_0|_v$ is given by $\beta = -v\alpha$ for $\alpha \in [0, u_c]$. It then follows, from (41), that

$$\bar{s}(v) > 0 \quad \forall v > v_c(u_c). \quad (42)$$

We now observe that, at any $v \geq 0$, the dynamical system (24) has a heteroclinic connection between \vec{e}_1 and \vec{e}_0 , in R^+ (which is unique, and is, in fact, contained in $\Omega_0 \subset R^+$) if and only if $\bar{s}(v) = 0$. It follows that since $\bar{s} : \overline{\mathbb{R}}^+ \rightarrow \mathbb{R}$ is a continuous and strictly monotone increasing function, which satisfies (39) and (42), then, for each $u_c \in (0, 1)$, there exists a unique $v^*(u_c) > 0$ such that

$$\bar{s}(v^*(u_c)) = 0, \quad (43)$$

while,

$$\bar{s}(v) < 0 \quad \forall v \in [0, v^*(u_c)), \quad \text{and} \quad \bar{s}(v) > 0 \quad \forall v \in (v^*(u_c), \infty). \quad (44)$$

We conclude that, for each $u_c \in (0, 1)$, QIVP has a PTW solution if and only if $v = v^*(u_c)(> 0)$ which we write as $u = U_T(y)$, $y \in \mathbb{R}$. Moreover, this PTW solution is unique. In addition, since the associated heteroclinic connection between \vec{e}_1 and \vec{e}_0 is contained in Ω_0 , then we conclude that $U_T : \mathbb{R} \rightarrow \mathbb{R}$ satisfies:

$$0 < U_T(y) < 1, \quad U_T'(y) < 0 \quad \forall y \in \mathbb{R}, \quad (45a)$$

with $U_T(0) = u_c$, and

$$U_T''(0^+) - U_T''(0^-) = -f_c^+, \quad (45b)$$

$$U_T(y) = u_c e^{-v^*(u_c)y} \quad \forall y \in \overline{\mathbb{R}}^+, \quad (45c)$$

$$U_T(y) \sim 1 - A_{-\infty} e^{\lambda_+(v^*(u_c))y} \quad \text{as } y \rightarrow -\infty, \quad (45d)$$

for some constant $A_{-\infty} > 0$ (depending upon $u_c \in (0, 1)$), and with the eigenvalue $\lambda_+(v)$ given in (29).

We next consider $u_c \in (0, 1)$ as a parameter, regarding v^* as a function of u_c , with $v^* : (0, 1) \rightarrow \mathbb{R}^+$ such that $v^* = v^*(u_c)$, and associated PTW solution $u = U_T(y, u_c)$ for $(y, u_c) \in \mathbb{R} \times (0, 1)$. We recall that the vector field $\vec{Q}(\alpha, \beta)$ is continuously differentiable on $(\alpha, \beta, v) \in ([0, u_c) \cup (u_c, 1]) \times \mathbb{R} \times \overline{\mathbb{R}}^+$, while the refraction vector (26) depends on $u_c \in (0, 1)$ and is continuous. It follows that on fixing $u_c^0 \in (0, 1)$, and taking $\varepsilon > 0$, then with $u_c = u_c^0$ and $v = v^*(u_c^0) - \varepsilon$, we have that $\bar{s}(v^*(u_c^0) - \varepsilon)|_{u_c = u_c^0} < 0$, where we have used Equation (43). Hence, there exists $\delta_\varepsilon^- > 0$, which depends on $\varepsilon > 0$, such that for all $u_c \in (u_c^0 - \delta_\varepsilon^-, u_c^0 + \delta_\varepsilon^-) = I_\varepsilon^-$, we have $\bar{s}(v^*(u_c^0) - \varepsilon)|_{u_c \in I_\varepsilon^-} < 0$. It follows that $v^*(u_c) > v^*(u_c^0) - \varepsilon$ for all $u_c \in I_\varepsilon^-$. Similarly, we establish that there exists $\delta_\varepsilon^+ > 0$, which depends on $\varepsilon > 0$, such that for all $u_c \in (u_c^0 - \delta_\varepsilon^+, u_c^0 + \delta_\varepsilon^+) = I_\varepsilon^+$, we have $\bar{s}(v^*(u_c^0) + \varepsilon)|_{u_c \in I_\varepsilon^+} > 0$. It follows that $v^*(u_c) < v^*(u_c^0) + \varepsilon$ for all $u_c \in I_\varepsilon^+$. We now set $\delta_\varepsilon = \min(\delta_\varepsilon^-, \delta_\varepsilon^+)$. Thus, for all $u_c \in (u_c^0 - \delta_\varepsilon, u_c^0 + \delta_\varepsilon) = I_\varepsilon$, we have $|v^*(u_c) - v^*(u_c^0)| < \varepsilon$. We conclude that $v^* : (0, 1) \rightarrow \mathbb{R}$ is continuous. In addition, we recall that

$$v^*(u_c) > 0 \quad \forall u_c \in (0, 1). \quad (46)$$

Next, let $u_c^0 \in (0, 1)$ and consider $S_1^+|_{(u_c^0, v^*(u_c^0))}$. It follows from the refraction vector (26) that there exists $\delta > 0$, such that on fixing $v = v^*(u_c^0)$, then for any $u_c \in (u_c^0, u_c^0 + \delta) = P_\delta$, the intersection point of $S_1^+|_{(u_c, v^*(u_c^0))}$ with the line $\alpha = u_c$ lies above the intersection point of the line $\beta = -v^*(u_c^0)\alpha$ with the line $\alpha = u_c$. Consequently, $\bar{s}(v^*(u_c^0))|_{u_c \in P_\delta} > 0$, from which we conclude that $v^*(u_c) < v^*(u_c^0)$ for all $u_c \in P_\delta$. Thus, $v^* : (0, 1) \rightarrow \mathbb{R}$ is locally decreasing, and continuous, and so $v^* : (0, 1) \rightarrow \mathbb{R}$ is strictly monotone decreasing. It then also follows from (46) that $v^*(u_c)$ has a finite nonnegative limit as $u_c \rightarrow 1^-$. Hence, $v^*(u_c) \rightarrow v_1^*$ as $u_c \rightarrow 1^-$, for some $v_1^* \geq 0$. When $(1 - u_c)$ is sufficiently small, the linearization theorem (see, for example, Ref. 25) guarantees that S_1^+ can be approximated in the region $(\alpha, \beta) \in [u_c, 1] \times \mathbb{R}^-$ by its linearized form at the equilibrium point

\vec{e}_1 ; it is then readily established that $v_1^* = 0$, and, moreover, that $v^*(u_c) \sim |f'_c(1)|^{\frac{1}{2}}(1 - u_c)$ as $u_c \rightarrow 1^-$. We now investigate $v^*(u_c)$ as $u_c \rightarrow 0^+$. To begin with we consider the dynamical system (24) when $u_c = 0$. In this case, the dynamical system (24) has a (unique) heteroclinic connection which connects \vec{e}_1 , as $y \rightarrow -\infty$, to \vec{e}_0 , as $y \rightarrow \infty$, if and only if $v \in [2, \infty)$, see, for example Refs. 5, 6, 9, 10. Moreover, $\bar{s}(v)|_{u_c=0} < 0$ for all $v \in [0, 2)$. From (26) and (27), it follows that S_1^+ depends continuously on $u_c \geq 0$. Thus, for $\varepsilon > 0$, there exists $\sigma_\varepsilon > 0$ such that for $u_c \in (0, \sigma_\varepsilon)$, then $\bar{s}(2 - \varepsilon)|_{u_c} < 0$. Therefore, from (43), we deduce that $v^*(u_c) > 2 - \varepsilon$ for all $u_c \in (0, \sigma_\varepsilon)$. However, it also follows from (26) and (27) that $\bar{s}(2)|_{u_c} > 0$ for all $u_c \in (0, 1)$. Thus, $v^*(u_c) < 2$ for all $u_c \in (0, 1)$. We conclude that, $2 - \varepsilon < v^*(u_c) < 2 \ \forall u_c \in (0, \sigma_\varepsilon)$. Since this holds for all $\varepsilon > 0$, we conclude immediately that $v^*(u_c)$ has limit 2 as $u_c \rightarrow 0^+$. We conclude that $v^* : (0, 1) \rightarrow \mathbb{R}$ is continuous and monotone decreasing, with

$$\lim_{u_c \rightarrow 1^-} v^*(u_c) = 0, \quad \lim_{u_c \rightarrow 0^+} v^*(u_c) = 2. \quad (47)$$

This completes the proof of Theorem 1. In the next two sections, we consider the structure of the PTW solutions in the limits $u_c \rightarrow 0^+$ and $u_c \rightarrow 1^-$, respectively.

4 | ASYMPTOTIC STRUCTURE OF THE PTW SOLUTION WHEN $u_c \rightarrow 0^+$

In this section, we investigate the detailed asymptotic form of $v^*(u_c)$ as $u_c \rightarrow 0^+$, in the small cut-off limit, via the method of matched asymptotic expansions. To that end, we write $u_c = \varepsilon$ with $0 < \varepsilon \ll 1$. It then follows from Theorem 1 that we may write,

$$v^*(\varepsilon) = 2 - \bar{v}(\varepsilon), \quad (48)$$

where now,

$$\bar{v}(\varepsilon) > 0 \quad \forall \varepsilon \in (0, 1), \quad \text{and} \quad \bar{v}(\varepsilon) = o(1) \quad \text{as} \quad \varepsilon \rightarrow 0^+. \quad (49)$$

With $U_T : \mathbb{R} \rightarrow \mathbb{R}$ being the associated PTW solution, then from (23),

$$U_{Tyy} + (2 - \bar{v}(\varepsilon))U_{Ty} + f(U_T) = 0, \quad y < 0, \quad (50a)$$

$$U_T(y) > \varepsilon \quad \forall y < 0, \quad (50b)$$

$$U_T(0) = \varepsilon, \quad U_{Ty}(0) = -(2 - \bar{v}(\varepsilon))\varepsilon, \quad (50c)$$

$$U_T(y) \rightarrow 1 \quad \text{as} \quad y \rightarrow -\infty. \quad (50d)$$

It is convenient, in what follows, to make a shift of origin by introducing the coordinate \bar{y} via

$$\bar{y} = \bar{y}_c(\varepsilon) + y,$$

where $\bar{y}_c(\varepsilon)$ is chosen so that (50) becomes,

$$U_{T\bar{y}\bar{y}} + (2 - \bar{v}(\varepsilon))U_{T\bar{y}} + f(U_T(\bar{y})) = 0, \quad \bar{y} < \bar{y}_c(\varepsilon), \quad (51a)$$

$$U_T(\bar{y}) > \varepsilon \quad \forall \bar{y} < \bar{y}_c(\varepsilon), \quad (51b)$$

$$U_T(\bar{y}_c(\varepsilon)) = \varepsilon, \quad U_{T\bar{y}}(\bar{y}_c(\varepsilon)) = -(2 - \bar{v}(\varepsilon))\varepsilon, \quad (51c)$$

$$U_T(\bar{y}) \rightarrow 1 \quad \text{as} \quad \bar{y} \rightarrow -\infty, \quad (51d)$$

with now the shift of origin fixing

$$U_T(0) = \frac{1}{2}. \quad (52)$$

It follows from (51) and (52) that

$$\bar{y}_c(\varepsilon) \rightarrow +\infty \quad \text{as} \quad \varepsilon \rightarrow 0^+. \quad (53)$$

Our objective is now to examine the boundary value problem (51) and (52) as $\varepsilon \rightarrow 0^+$, and, in particular, to determine the asymptotic structure of $\bar{v}(\varepsilon)$ as $\varepsilon \rightarrow 0^+$. Anticipating the requirement of outer regions, we begin in an inner region when $\bar{y} = O(1)$ and $U_T = O(1)$ as $\varepsilon \rightarrow 0^+$, and we label this as region **I**. In region **I**, we thus expand as

$$U_T(\bar{y}; \varepsilon) = U_m(\bar{y}) + O(\bar{v}(\varepsilon)) \quad \text{as} \quad \varepsilon \rightarrow 0^+, \quad (54)$$

with $\bar{y} = O(1)$. On substitution from (54) into (51) and (52), and using (53), we obtain the leading order problem as

$$U_{m\bar{y}\bar{y}} + 2U_{m\bar{y}} + f(U_m) = 0, \quad -\infty < \bar{y} < \infty, \quad (55a)$$

$$U_m(\bar{y}) > 0, \quad -\infty < \bar{y} < \infty, \quad (55b)$$

$$U_m(\bar{y}) \rightarrow \begin{cases} 1, & \text{as } \bar{y} \rightarrow -\infty, \\ 0, & \text{as } \bar{y} \rightarrow \infty, \end{cases} \quad (55c)$$

$$U_m(0) = \frac{1}{2}. \quad (55d)$$

The leading order problem is immediately recognized as the boundary value problem (23) for the permanent form traveling wave solution to the corresponding KPP problem without cut-off ($\varepsilon = 0$). Let $U_m : \mathbb{R} \rightarrow \mathbb{R}$ be the unique solution to (55). For use in what follows, we recall (7) with higher-order corrections given by

$$U_m(\bar{y}) = \begin{cases} (A_\infty \bar{y} + B_\infty) e^{-\bar{y}} + O(\bar{y}^2 e^{-2\bar{y}}), & \text{as } \bar{y} \rightarrow \infty, \\ 1 - A_{-\infty} e^{\gamma \bar{y}} + O(e^{2\gamma \bar{y}}), & \text{as } \bar{y} \rightarrow -\infty, \end{cases} \quad (56)$$

where $\gamma = -1 + \sqrt{1 + |f'(1)|}$ (> 0). On proceeding to $O(\bar{v}(\varepsilon))$ in region **I**, we observe that the inner region expansion (54) becomes non-uniform when $|\bar{y}| \gg 1$, and in particular when $(-\bar{y}) = O(\bar{v}(\varepsilon)^{-\frac{1}{2}})$ and $\bar{y} = O(\bar{v}(\varepsilon)^{-\frac{1}{2}})$. Therefore, to complete the asymptotic structure of the solution to (51) as $\varepsilon \rightarrow 0^+$, we must introduce two outer regions, namely, region **II**⁺ when $\bar{y} = O(\bar{v}(\varepsilon)^{-\frac{1}{2}})$ and region **II**[−] when $(-\bar{y}) = O(\bar{v}(\varepsilon)^{-\frac{1}{2}})$. In this context, for any variable λ , we will henceforth write $\lambda = O(1) > 0$ as $\lambda = O(1)^+$, and correspondingly, $\lambda = O(1) < 0$ as $\lambda = O(1)^−$. We begin in region **II**[−]. To formalize region **II**[−], we introduce the scaled variable,

$$\hat{y} = \bar{v}(\varepsilon)^{\frac{1}{2}} \bar{y}, \quad (57)$$

so that $\hat{y} = O(1)^−$ in region **II**[−] as $\varepsilon \rightarrow 0^+$. It then follows from (54) and (56) that

$$U_T(\hat{y}; \varepsilon) = 1 - O\left(e^{-\bar{v}(\varepsilon)^{-\frac{1}{2}}}\right), \quad (58)$$

as $\varepsilon \rightarrow 0^+$ in region **II**[−]. It is then straightforward to develop an exponential expansion in region **II**[−], which, after matching (following the Van Dyke matching principle²⁶) with region **I**, via (54) and (56), gives the outer expansion in region **II**[−] as,

$$U_T(\hat{y}; \varepsilon) = 1 - A_{-\infty} \exp\left[\gamma \bar{v}(\varepsilon)^{-\frac{1}{2}} (1 + O(\bar{v}(\varepsilon))) \hat{y}\right] + O\left(\exp\left[2\gamma \bar{v}(\varepsilon)^{-\frac{1}{2}} (1 + O(\bar{v}(\varepsilon))) \hat{y}\right]\right), \quad (59)$$

as $\varepsilon \rightarrow 0^+$ with $\hat{y} = O(1)^−$. Thus, the solution in region **II**[−] is at this order unaffected by the cut-off. We now proceed to region **II**⁺, where $\hat{y} = O(1)^+$ as $\varepsilon \rightarrow 0^+$. It is within this region that the conditions at $\bar{y} = \bar{y}_c(\varepsilon)$ must be satisfied, which then requires $\bar{y}_c(\varepsilon) = O(\bar{v}(\varepsilon)^{-\frac{1}{2}})$ as $\varepsilon \rightarrow 0^+$, which is consistent with (53). Thus, we write

$$\bar{y}_c(\varepsilon) = \bar{v}(\varepsilon)^{-\frac{1}{2}} \hat{y}_c(\varepsilon), \quad (60)$$

so that now,

$$\hat{y}_c(\varepsilon) = O(1)^+ \quad \text{as } \varepsilon \rightarrow 0^+. \quad (61)$$

In region \mathbf{II}^+ , it follows from (54) and (56) that

$$U_T(\hat{y}; \varepsilon) = O\left(\bar{v}(\varepsilon)^{-\frac{1}{2}} e^{-\bar{v}(\varepsilon)^{-\frac{1}{2}}}\right),$$

as $\varepsilon \rightarrow 0^+$. Again, it is then straightforward to develop an exponential expansion in region \mathbf{II}^+ , which, after matching with region \mathbf{I} , via (54) and (56), gives the outer expansion in region \mathbf{II}^+ as,

$$\begin{aligned} U_T(\hat{y}; \varepsilon) = & \left(A_\infty \bar{v}(\varepsilon)^{-\frac{1}{2}} \sin(\hat{y}(1 + O(\bar{v}(\varepsilon)))) + B_\infty \cos(\hat{y}(1 + O(\bar{v}(\varepsilon)))) \right) \\ & \times \exp\left[-\bar{v}(\varepsilon)^{-\frac{1}{2}}(1 + O(\bar{v}(\varepsilon)))\hat{y}\right] + O\left(\exp\left[-2\bar{v}(\varepsilon)^{-\frac{1}{2}}(1 + O(\bar{v}(\varepsilon)))\hat{y}\right]\right), \end{aligned} \quad (62)$$

as $\varepsilon \rightarrow 0^+$ with $\hat{y} = O(1)^+$. It now remains to apply conditions (51b), and (51c) to (62). In the outer region \mathbf{II}^+ , these conditions become,

$$U_T(\hat{y}; \varepsilon) > \varepsilon \quad \forall O\left(\bar{v}(\varepsilon)^{-\frac{1}{2}}\right)^+ < \hat{y} < \hat{y}_c(\varepsilon), \quad (63a)$$

$$U_T(\hat{y}_c(\varepsilon); \varepsilon) = \varepsilon, \quad U_{T\hat{y}}(\hat{y}_c(\varepsilon); \varepsilon) = -\varepsilon \bar{v}(\varepsilon)^{-\frac{1}{2}}(2 - \bar{v}(\varepsilon)). \quad (63b)$$

We now turn to conditions (63b). It is convenient to first eliminate ε explicitly to give,

$$U_{T\hat{y}}(\hat{y}_c(\varepsilon); \varepsilon) = -\bar{v}(\varepsilon)^{-\frac{1}{2}}(2 - \bar{v}(\varepsilon))U_T(\hat{y}_c(\varepsilon); \varepsilon), \quad (64)$$

which replaces (63b). On substitution from (62) into (64) and expanding, using (49), (50), and (61), we obtain,

$$A_\infty \sin \omega = -\bar{v}(\varepsilon)^{\frac{1}{2}}(A_\infty + B_\infty) \cos \omega, \quad \omega = \hat{y}_c(\varepsilon)(1 + O(\bar{v}(\varepsilon))), \quad (65)$$

as $\varepsilon \rightarrow 0^+$. Following (61) and (65), we now expand,

$$\hat{y}_c(\varepsilon) = \hat{y}_c^0 + \hat{y}_c^1 \bar{v}(\varepsilon)^{\frac{1}{2}} + O(\bar{v}(\varepsilon)), \quad (66)$$

as $\varepsilon \rightarrow 0^+$, with the constants $\hat{y}_c^0 (> 0)$ and \hat{y}_c^1 to be determined. On substitution from (66) into (65), we obtain, at $O(1)$,

$$A_\infty \sin \hat{y}_c^0 = 0.$$

Since $A_\infty > 0$, then we must have (recalling $\hat{y}_c^0 > 0$) $\hat{y}_c^0 = k\pi$, for some $k \in \mathbb{N}$. However, condition (63a), with (62), then requires $k = 1$, and so

$$\hat{y}_c^0 = \pi. \quad (67)$$

Proceeding to $O(\bar{v}(\varepsilon)^{\frac{1}{2}})$, we find that, on using (67),

$$\hat{y}_c^1 = -\frac{(A_\infty + B_\infty)}{A_\infty}. \quad (68)$$

Thus, via (66), (67), and (68) we have,

$$\hat{y}_c(\varepsilon) = \pi - \frac{(A_\infty + B_\infty)}{A_\infty} \bar{v}(\varepsilon)^{\frac{1}{2}} + O(\bar{v}(\varepsilon)), \quad (69)$$

as $\varepsilon \rightarrow 0^+$. It remains to apply the first condition in (63b). On using (62) and (69), the first condition in (63b) becomes

$$\ln \varepsilon = -\frac{\pi}{\bar{v}(\varepsilon)^{\frac{1}{2}}} + \left(\frac{(A_\infty + B_\infty)}{A_\infty} + \ln A_\infty \right) + O(\bar{v}(\varepsilon)^{\frac{1}{2}}), \quad (70)$$

as $\varepsilon \rightarrow 0^+$. A rearrangement of (70) then gives,

$$\bar{v}(\varepsilon) = \frac{\pi^2}{(\ln \varepsilon)^2} + \frac{2\pi^2((A_\infty + B_\infty)A_\infty^{-1} + \ln A_\infty)}{(\ln \varepsilon)^3} + O\left(\frac{1}{(\ln \varepsilon^4)}\right), \quad (71)$$

as $\varepsilon \rightarrow 0^+$. It then follows from (69) and (71) that,

$$\hat{y}_c(\varepsilon) = \pi + \frac{(A_\infty + B_\infty)\pi}{A_\infty} \frac{1}{\ln \varepsilon} + O\left(\frac{1}{(\ln \varepsilon^2)}\right), \quad (72)$$

as $\varepsilon \rightarrow 0^+$. Finally, via (48) and (71), we can construct $v^*(\varepsilon)$ as

$$v^*(\varepsilon) = 2 - \frac{\pi^2}{(\ln \varepsilon)^2} - \frac{2\pi^2((A_\infty + B_\infty)A_\infty^{-1} + \ln A_\infty)}{(\ln \varepsilon)^3} + O\left(\frac{1}{(\ln \varepsilon^4)}\right), \quad (73)$$

as $\varepsilon \rightarrow 0^+$. For completeness, we give a schematic diagram of the asymptotic structure for $U_T(\bar{y}; \varepsilon)$ in terms of the coordinate \bar{y} as $\varepsilon \rightarrow 0^+$ in Figure 5. Returning to (73) we observe that the approximation is decreasing in ε as $\varepsilon \rightarrow 0^+$, and is in full accord with the rigorous results established in Theorem 1. We see immediately that the approximation derived here, agrees in the first two terms with prediction (9) that Brunet and Derrida¹⁴ first obtained and its third term is consistent with the order of the error term in (10) that Dumortier et al²⁰ derived. However, the method of matched asymptotic expansions has enabled us to obtain the next correction term in (73), and higher-order terms could be obtained by systematically following this approach (of course it may also be possible to obtain the third- and higher-order terms via extending the approach of Ref. 20). In fact, we may continue the expansion in each region to next order, and after matching, we can readily

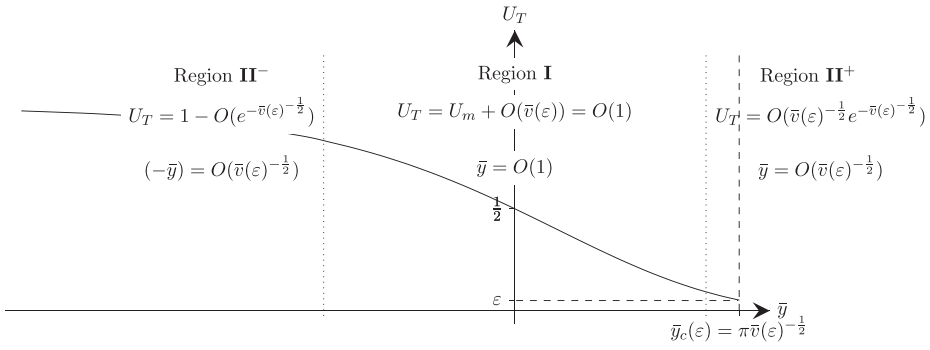


FIGURE 5 A schematic diagram of the asymptotic structure of $U_T(\bar{y}; \varepsilon)$ as $u_c \rightarrow 0^+$

obtain that the higher-order correction to (73) is given by

$$\begin{aligned} v^*(\varepsilon) = 2 - \frac{\pi^2}{(\ln \varepsilon)^2} - \frac{2\pi^2((A_\infty + B_\infty)A_\infty^{-1} + \ln A_\infty)}{(\ln u_c)^3} \\ + \frac{3\pi^2\left(\frac{1}{4}\pi^2 - ((A_\infty + B_\infty)A_\infty^{-1} + \ln A_\infty)^2\right)}{(\ln \varepsilon)^4} + O\left(\frac{1}{(\ln \varepsilon)^5}\right), \end{aligned} \quad (74)$$

as $\varepsilon \rightarrow 0^+$. For brevity, we do not provide a derivation to (74). We now consider the asymptotic structure of the PTW solution to QIVP as $u_c \rightarrow 1^-$.

5 | ASYMPTOTIC STRUCTURE OF THE PTW SOLUTION WHEN $u_c \rightarrow 1^-$

In this section, we investigate the asymptotic form of $v^*(u_c)$ in the large cut-off limit $u_c \rightarrow 1^-$. To this end, we write $u_c = 1 - \delta$ with $0 < \delta \ll 1$. Theorem 1 guarantees the existence and uniqueness of a PTW solution, whose speed $v^*(\delta) = o(1)$ as $\delta \rightarrow 0^+$. In this case, it is most convenient to consider the problem in the (α, β) phase plane corresponding to the phase path representing the PTW when $u_c = 1 - \delta$ and $v = v^*(\delta)$. Via (27), (29), (30), and (31), this is given by the phase path $\beta = \beta(\alpha; \delta)$, which satisfies the boundary value problem

$$\frac{d\beta}{d\alpha} = -v^*(\delta) - \frac{f(\alpha)}{\beta}, \quad \alpha \in (1 - \delta, 1), \quad (75a)$$

$$\beta(\alpha; \delta) \sim -\lambda_+(v^*(\delta))(1 - \alpha) \quad \text{as } \alpha \rightarrow 1^-, \quad (75b)$$

$$\beta(1 - \delta; \delta) = -v^*(\delta)(1 - \delta). \quad (75c)$$

We now examine the boundary value problem (75) as $\delta \rightarrow 0^+$. Since $v^*(\delta) = o(1)$ as $\delta \rightarrow 0^+$, we expand $\lambda_+(v^*(\delta))$, via (29), which determines that $\lambda_+(v^*(\delta)) = O(1)$ as $\delta \rightarrow 0^+$. It follows from the boundary condition (75b), that $\beta = O(\delta)$ as $\delta \rightarrow 0^+$. We therefore introduce the following rescalings:

$$\beta = \delta Y, \quad \alpha = 1 - \delta X, \quad (76)$$

with $Y, X = O(1)$ as $\delta \rightarrow 0^+$. The form of the boundary condition (75c) then necessitates that $v^*(\delta) = O(\delta)$ as $\delta \rightarrow 0^+$. Thus, we write

$$v^*(\delta) = \delta V(\delta), \quad (77)$$

where $V(\delta) = O(1)$ as $\delta \rightarrow 0^+$. These rescalings transform the boundary value problem (75) into

$$\frac{dY}{dX} = \delta V(\delta) + \frac{f(1 - \delta X)}{\delta Y}, \quad X \in (0, 1), \quad (78a)$$

$$Y(X; \delta) \sim -\lambda_+(\delta V(\delta))X \quad \text{as } X \rightarrow 0^+, \quad (78b)$$

$$Y(1; \delta) = -V(\delta)(1 - \delta). \quad (78c)$$

We now expand $Y(X; \delta)$ and $V(\delta)$ according to,

$$Y(X; \delta) = Y_0(X) + \delta Y_1(X) + o(\delta), \quad X \in [0, 1], \quad (79a)$$

$$V(\delta) = V_0 + \delta V_1 + o(\delta), \quad (79b)$$

as $\delta \rightarrow 0^+$. Substituting the expansions from (79) into the boundary value problem (78) and expanding, at $O(1)$, we obtain the following boundary value problem for $Y_0(X)$, namely,

$$\frac{dY_0}{dX} = -f'(1) \frac{X}{Y_0}, \quad X \in (0, 1), \quad (80a)$$

$$Y_0(X) \sim -|f'(1)|^{\frac{1}{2}} X \quad \text{as } X \rightarrow 0^+, \quad (80b)$$

$$Y_0(1) = -V_0. \quad (80c)$$

The general solution to (80a) is $Y_0^2(X) = c_1 - f'(1)X^2$, for $X \in [0, 1]$, where c_1 is an arbitrary constant of integration. Applying the boundary condition (80b) determines $c_1 = 0$. Therefore,

$$Y_0(X) = -|f'(1)|^{\frac{1}{2}} X, \quad X \in [0, 1]. \quad (81)$$

Application of the boundary condition (80c) then determines

$$V_0 = |f'(1)|^{\frac{1}{2}}. \quad (82)$$

At $O(\delta)$, we obtain the following boundary value problem for $Y_1(X)$, namely,

$$\frac{dY_1}{dX} - \frac{Y_1}{Y_0(X)^2} f'(1)X = V_0 + \frac{1}{2} f''(1) \frac{X^2}{Y_0(X)}, \quad X \in (0, 1), \quad (83a)$$

$$Y_1(X) \sim \frac{1}{2} V_0 X \quad \text{as } X \rightarrow 0^+, \quad (83b)$$

$$Y_1(1) = V_0 - V_1. \quad (83c)$$

On substituting $Y_0(X)$, given by (81), into Equation (83a) and solving, we find that the general solution is

$$Y_1(X) = \frac{1}{2} V_0 X - \frac{1}{6} \frac{f''(1)}{|f'(1)|^{\frac{1}{2}}} X^2 + \frac{c_2}{X}, \quad X \in (0, 1], \quad (84)$$

where c_2 is an arbitrary constant of integration. From the boundary condition (83b), $Y_1(X)$ remains bounded as $X \rightarrow 0^+$. Therefore, we require $c_2 = 0$. Thus, we obtain the solution for $Y_1(X)$ as

$$Y_1(X) = \frac{1}{6} |f'(1)|^{\frac{1}{2}} X \left(3 - \frac{f''(1)}{|f'(1)|} X \right), \quad X \in [0, 1]. \quad (85)$$

Finally, an application of the boundary condition (86) determines

$$V_1 = \frac{1}{6} |f'(1)|^{\frac{1}{2}} \left(3 + \frac{f''(1)}{|f'(1)|} \right). \quad (86)$$

On collecting expressions (79a), (81), and (85), we have established that

$$\begin{aligned} Y(X; \delta) = & -|f'(1)|^{\frac{1}{2}} X + \frac{1}{6} \delta |f'(1)|^{\frac{1}{2}} X \left(3 - \frac{f''(1)}{|f'(1)|} X \right) \\ & + o(\delta) \quad \text{as } \delta \rightarrow 0^+, \end{aligned} \quad (87)$$

uniformly for $X \in [0, 1]$. Similarly, on collecting expressions (79b), (82), and (86), we obtain,

$$V(\delta) = |f'(1)|^{\frac{1}{2}} + \frac{1}{6} \delta |f'(1)|^{\frac{1}{2}} \left(3 + \frac{f''(1)}{|f'(1)|} \right) + o(\delta) \quad \text{as } \delta \rightarrow 0^+. \quad (88)$$

We use (76) to express the PTW solution to QIVP in terms of the cut-off u_c as

$$\begin{aligned} \beta(\alpha) = & -\frac{1}{2}|f'(1)|^{\frac{1}{2}}(1+u_c)(1-\alpha) - \frac{1}{6}\frac{f''(1)}{|f'(1)|^{\frac{1}{2}}}(1-\alpha)^2 \\ & + o((1-u_c)^2) \quad \text{as } u_c \rightarrow 1^-, \end{aligned} \quad (89)$$

with $\alpha \in [u_c, 1]$. Its speed of propagation, via (77) and (88), is given by

$$\begin{aligned} v^*(u_c) = & (1-u_c)|f'(1)|^{\frac{1}{2}} + \frac{1}{6}(1-u_c)^2|f'(1)|^{\frac{1}{2}}\left(3 + \frac{f''(1)}{|f'(1)|}\right) \\ & + o((1-u_c)^2) \quad \text{as } u_c \rightarrow 1^-. \end{aligned} \quad (90)$$

In the next section, we consider the specific case of a cut-off Fisher reaction, determining $U_T : \mathbb{R} \rightarrow \mathbb{R}$ and $v^* : (0, 1) \rightarrow \mathbb{R}$ via numerical integration.

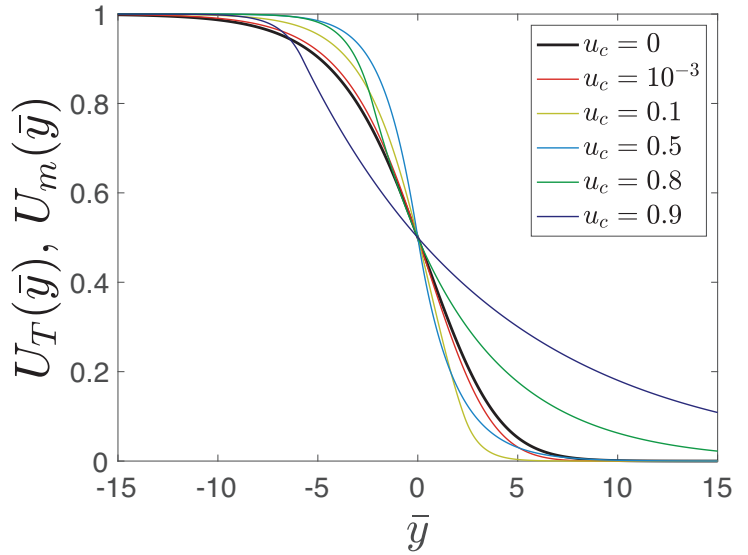
6 | NUMERICAL EXAMPLE

We now focus on the particular case of the cut-off Fisher reaction function, namely, (8d) with (3a). We obtain numerical approximations of the speed $v^*(u_c)$ and PTW solutions $U_T : \mathbb{R} \rightarrow \mathbb{R}$ for a range of values of cut-off u_c . This is achieved by solving (11) numerically over an interval $y \in [0, M]$ for $M \in \mathbb{R}^+$ using the MATLAB initial value solver ode45 where $U_T(0) = u_c$. As “initial” condition we employ (30) with 7(c) to approximate the unstable manifold near the unstable fixed point $(U_T, U'_T) = (1, 0)$, taking $U_T(0) = 1 - \epsilon$ and $U'_T(0) = -\lambda_+(v)\epsilon$ where $\epsilon = 10^{-12}$ and prescribe an absolute and relative tolerance of 10^{-13} . The value of v in the second initial condition is not known *a priori*. We therefore build the initial value solver into a shooting type algorithm, for which we guess the value of v , integrate (11) to obtain $U'_T(0)$, and then compare $U'_T(0)$ to the target value $-vu_c$. The value of v is then modified using the bisection method and this integration procedure is iterated until the absolute error satisfies $|U'_T(0) + vu_c| < 10^{-13}$. We start with a value of u_c close to 1 and take $v = 2$ as initial guess to begin the iteration which leads to the solution with speed $v^*(u_c)$. We then iterate over decreasing values of u_c using the previously determined value of $v^*(u_c)$ as an initial guess to find the next solution.

It is also useful to obtain a numerical approximation of the permanent traveling wave solution U_m for the Fisher reaction function (3a) in the absence of a cut-off. This is readily achieved by solving (6) numerically over an interval $y \in [0, N]$ for $N \in \mathbb{R}^+$ once more using the Matlab initial value solver ode45. As “initial condition” we employ (7b) and (7c) to approximate the unstable manifold near the unstable fixed point $(U_m, U'_m) = (1, 0)$, taking $v = 2$, $U_m(0) = 1 - \epsilon$, and $U'_m(0) = (\sqrt{2} - 1)\epsilon$, where $\epsilon = 10^{-12}$ and prescribe an absolute and relative tolerance of 10^{-13} . We then determine that value of y for which U_m is equal to $1/2$ and then perform a coordinate shift to the origin.

Figure 6 contrasts the behavior of U_T against U_m . A direct comparison between U_T and U_m is achieved when U_T and U_m are expressed in terms of \bar{y} so that $U_T(0) = U_m(0) \approx 0.5$. For small values of u_c , $U_T(\bar{y})$ is close to $U_m(\bar{y})$ in agreement with the asymptotic theory of Section 4. As u_c increases, we observe a strong departure of $U_T(\bar{y})$ from $U_m(\bar{y})$ with the slope of $U_T(\bar{y})$ at $\bar{y} = 0$ becoming increasingly steep until it reaches a maximum at $u_c = 0.5$. Beyond this value,

FIGURE 6 Permanent form traveling wave solutions $U_T(\bar{y})$ computed numerically as a function of \bar{y} for selected cut-off values u_c . These are compared against the traveling wave solution $U_m(\bar{y})$ obtained in the absence of a cut-off (corresponding to $u_c = 0$)



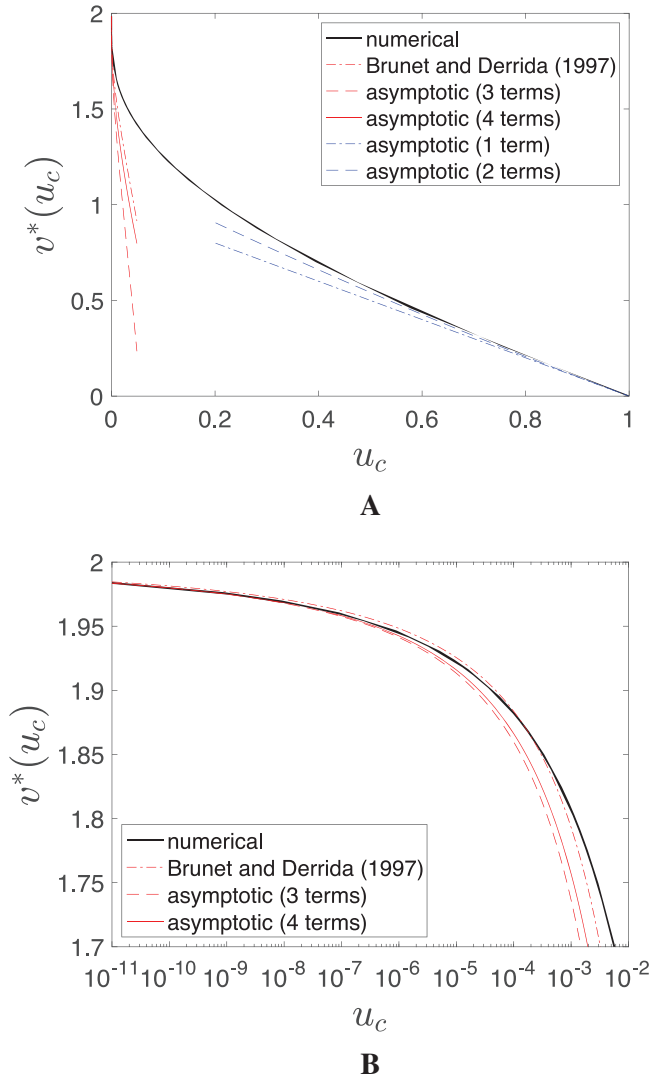
the slope satisfies $U_T'(0) = -v^*(u_c)/2$ and therefore becomes increasingly gentle with u_c (since $v^*(u_c)$ decreases with u_c). When u_c approaches 1, $U_T(\bar{y})$ is in full agreement with the asymptotic prediction, derived from (89) (not shown).

Figure 7 examines the behavior of the speed $v^*(u_c)$ and compares it to the various asymptotic expansions obtained as $u_c \rightarrow 0^+$ and $u_c \rightarrow 1^-$ (as derived from (74) and (90), respectively). The asymptotic expansion (74) obtained for $u_c \rightarrow 0^+$ relies on the global constants A_∞ and B_∞ associated with the leading edge behavior of $U_m(\bar{y})$ (see (56)). We determine the values of A_∞ and B_∞ by performing a least-squares polynomial fit to the computed $U_m(\bar{y})e^{\bar{y}}$ for $\bar{y} \gtrsim 10$ from where we obtain to a very good approximation the linear polynomial fit with

$$A_\infty \approx 3.5 \quad \text{and} \quad B_\infty \approx -11.2. \quad (91)$$

Figure 7A demonstrates that the two-term asymptotic expansion of $v^*(u_c)$ as $u_c \rightarrow 1^-$ accurately captures the speed $v^*(u_c)$ for a wide range of values given by $0.4 \lesssim u_c < 1$ (when $u_c = 0.4$, $\delta = 1 - u_c = 0.6$ associated with expansions (79) is no longer small). Figure 7B focusses on the behavior of the speed obtained for smaller values of u_c . It shows that the curve representing the two-term expansion based on retaining two terms in (74) and corresponding to the asymptotics that Brunet and Derrida¹⁴ first obtained crosses the numerically computed curve representing $v^*(u_c)$ at $u_c \approx 10^{-4}$ and has a monotonic approach to this curve for smaller values of u_c . We therefore anticipate that this two-term expansion only becomes genuinely asymptotic for values of $u_c \ll 10^{-4}$. This implies that any comparison of the three-term and four-term expansions based on retaining three and four terms in (74), respectively, should only be considered for $u_c \ll 10^{-4}$. With this in mind, the logarithmic corrections included in our three- and four-term expansions are an improvement over the two-term expansion for $u_c \ll 10^{-4}$ and a reasonable approximation to $v^*(u_c)$ with acceptable accuracy, less than 10 percentage error, for $u_c \lesssim 8 \times 10^{-3}$ and for $u_c \lesssim 4 \times 10^{-3}$, respectively. However, the higher-order logarithmic corrections that they neglect are significant for larger values of u_c .

FIGURE 7 A, Propagation speed $v^*(u_c)$ computed numerically as a function of the cut-off value u_c for the particular case of the cut-off Fisher reaction function (8d) with (3a). Comparison against the asymptotic expansions for $v^*(u_c)$ obtained as $u_c \rightarrow 1^-$ based on retaining one and two terms in (90) and as $u_c \rightarrow 0^+$ based on retaining two, three, and four terms in (74). The two-term asymptotic expansion of $v^*(u_c)$ as $u_c \rightarrow 0^+$ corresponds to prediction (9) first obtained in Brunet and Derrida¹⁴. B, Same as (A) but focusing on smaller values of u_c



7 | CONCLUSIONS

In this paper, we have considered a canonical evolution problem for a reaction-diffusion process when the reaction function is of standard KPP-type, but experiences a cut-off in the reaction rate below the normalized cut-off concentration $u_c \in (0, 1)$. We have formulated this evolution problem in terms of the moving boundary initial-boundary value problem QIVP. In Section 2, we have obtained some very general results concerning the solution to QIVP. In particular, these general results indicate that in the large time, as $t \rightarrow \infty$, the solution to QIVP will involve the propagation of an advancing nonnegative permanent form traveling wave, effecting the transition from the unreacting state $u = 0$ (ahead of the wave front) to the fully reacted state $u = 1$ (at the rear of the wave front). With this in mind, this paper has concentrated on examining the existence of permanent form traveling wave solutions to QIVP with propagation speed $v \geq 0$, referred to as PTW solutions. In Section 3, we have used a phase plane analysis of the nonlinear boundary value problem (23) to establish that (a) for each $u_c \in (0, 1)$, then QIVP has a unique PTW

solution, with propagation speed $v = v^*(u_c) > 0$ and (b) $v^* : (0, 1) \rightarrow \mathbb{R}^+$ is continuous and monotone decreasing, with $v^*(u_c) \rightarrow 0^+$ as $u_c \rightarrow 1^-$, and $v^*(u_c) \rightarrow 2^-$ as $u_c \rightarrow 0^+$. It should be noted that 2 is the minimum propagation speed of permanent form traveling wave solutions for the related KPP-type function in the absence of cut-off. In Section 4, we have developed asymptotic methods to determine the asymptotic forms of $v^*(u_c)$ as $u_c \rightarrow 0^+$ and $u_c \rightarrow 1^-$. The first limit was previously considered by Brunet and Derrida¹⁴ and Dumortier et al²⁰. The latter employed geometric desingularization to systematically determine the order of the error in Ref. 14. We have here used matched asymptotics expansions on the direct problem (23) to obtain higher-order corrections in a systematic manner. We show that these are controlled by the detailed structure ahead of the wave front solution traveling with speed 2 for the related KPP problem obtained in the absence of a cut-off. The second limit of $u_c \rightarrow 1^-$ is motivated by applications in combustion¹. In this limit, the asymptotic behavior is obtained via the use of regular asymptotic expansions in the phase plane.

We anticipate that the approach developed in this paper, for considering PTW solutions to QIVP, will be readily adaptable to corresponding problems, when the cut-off KPP-type reaction considered here is replaced by a broader class of cut-off reaction functions, such as those considered in Refs. 15, 20–23. In comparing the PTW theory for the cut-off KPP-type reaction function studied here, and its associated KPP-type reaction function without cut-off, we make the observation that, in the absence of cut-off, a PTW solution exists for each propagation speed $v \in [2, \infty)$, while at each fixed cut-off value $u_c \in (0, 1)$, a PTW solution exists only at the single propagation speed $v = v^*(u_c)$, with $0 < v^*(u_c) < 2$; this observation has been made previously in Ref. 20, although restricted to sufficiently small cut-off values u_c . This will have implications for the development of PTW solutions as large- t structures in QIVP, with more general classes of initial data. In the companion paper, we consider the evolution problem QIVP in more detail. Specifically we establish that, as $t \rightarrow \infty$, the solution to QIVP does indeed involve the formation of the PTW solution considered in this paper, and we give the detailed asymptotic structure of the solution to QIVP as $t \rightarrow \infty$.

Finally, it is interesting to contrast our results with results obtained for a related problem, the stochastic KPP equation

$$u_t = u_{xx} + f(u) + (\hat{u}_c f(u))^{1/2} \dot{W}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad (92)$$

with $u(x, 0) = u_0(x)$, where \dot{W} is a standard space-time white-noise. Similarly to the cut-off KPP equation (8), Equation (92) arises as a continuum approximation to (microscopic) interacting particle systems. In particular, for a Fisher reaction function (3a), there is an exact relationship between this problem and discrete systems of particles which undergo a birth-coagulation type of reaction in addition to diffusion^{27,28}. Rigorous results have been derived for this model too^{29,30}, establishing that the average speed of the random traveling wave solutions of (92) is, in the small- \hat{u}_c or, weak noise limit, given by

$$v_s(\hat{u}_c) = 2 - \frac{\pi^2}{(\ln \hat{u}_c)^2} + O\left(\frac{\ln |\ln \hat{u}_c|}{|\ln \hat{u}_c|^3}\right), \quad \text{as } \hat{u}_c \rightarrow 0^+. \quad (93)$$

Thus, taking $\hat{u}_c = u_c$, the difference between (93) and the speed of the PTW solution of the cut-off KPP model (8) only arises in the third term of the asymptotic expansion of $v_s(\hat{u}_c)$ and $v^*(u_c)$ as $u_c \rightarrow 0^+$, a conjecture that was initially made by Brunet and Derrida^{14,31}. The two models behave very differently when \hat{u}_c can no longer be regarded as small, as might be anticipated. In the

large- \hat{u}_c or, strong noise limit^{28,32}, find that

$$v_s(\hat{u}_c) \sim \sqrt{\frac{1}{\hat{u}_c}}, \quad \text{as } \hat{u}_c \rightarrow \infty. \quad (94)$$

The behavior in this limit should be contrasted against expression (90) obtained for $u_c \rightarrow 1^-$. A comparison suggests that \hat{u}_c and u_c may in this case be related according to $\hat{u}_c \sim 1/(1 - u_c)^2$ as $u_c \rightarrow 1^-$. It would be interesting to extend this comparison to arbitrary u_c .

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